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ORTHOGONALITY CONDITION IN OFDM WITH FILTERING

The modulated waveform corresponding to the k_ith subchannel is given by

$$S_i(n) = a(k_i)exp(\frac{j2\pi k_i n}{N})$$

The modulated waveform corresponding to the k_jth subchannel is given by

$$S_j(n) = a(k_j)exp(\frac{j2\pi k_j n}{N})$$

Where k_i and k_j are the subchannel numbers and N is the IFFT size and $a(k_i)$ and $a(k_j)$ are the modulation symbols corresponding to these subchannels.

The inner product of the modulated waveforms is given by

$$\sum_{n=0}^{N-1} S_i(n)S_j^*(n)$$

$$= \sum_{n=0}^{N-1} a(k_i)a^*(k_j)exp(\frac{j2\pi(k_i - k_j)n}{N})$$

If $k_i = k_j$, the above exponential term becomes $e^{j2\pi 0n} = 1$ which eventually becomes a summation of 1s.
If $k_i \neq k_j$ the above expression represents the sum of a finite geometric series.
The sum of a finite geometric series $1, r, r^2, r^3, ...r^{N-1}$ is given by $\frac{1-r^N}{1-r}$
Substituting for $r = exp(\frac{j2\pi(k_i-k_j)}{N})$, we get

$$a(k_i)a^*(k_j)\left\{\frac{1 - exp(\frac{j2\pi(k_i-k_j)N}{N})}{1 - exp(\frac{j2\pi(k_i-k_j)}{N})}\right\}$$

$$= a(k_i)a^*(k_j)\left\{\frac{1 - exp(j2\pi(k_i - k_j))}{1 - exp(\frac{j2\pi(k_i-k_j)}{N})}\right\}$$

Since k_i and k_j are two integers, $exp(j2\pi(k_i - k_j)) = 1$

$$= a(k_i)a^*(k_j)\left\{\frac{1 - 1}{1 - exp(\frac{j2\pi(k_i-k_j)}{N})}\right\} = 0$$

Hence, we can conclude the following.

$$\sum_{n=0}^{N-1} S_i(n) S_j^*(n) = \begin{cases} 0, & \text{if } k_i \neq k_j \\ a(k_i) a^*(k_j) N & \text{if } k_i = k_j \end{cases} \tag{1}$$

The above proves the orthogonality of the waveforms. The orthogonality requires the condition that the symbol duration is the reciprocal of the subcarrier spacing.

Applying FIR filtering on the modulated waveforms $S_i(n)$ and $S_j(n)$ result in the following

$$Y_i(n) = S_i(n) * h_i(n)$$

$$= \sum_{l=0}^{L-1} a(k_i) exp(\frac{j2\pi k_i (n-l)}{N}) h_i(l)$$

$$Y_j(n) = S_j(n) * h_j(n)$$

$$= \sum_{m=0}^{L-1} a(k_j) exp(\frac{j2\pi k_j (n-m)}{N}) h_j(m)$$

The inner product of the filtered waveforms is given by

$$\sum_{n=0}^{N-1} Y_i(n) Y_j^*(n)$$

$$= \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} a(k_i) exp(\frac{j2\pi k_i (n-l)}{N}) h_i(l) \sum_{m=0}^{L-1} a^*(k_j) exp(-\frac{j2\pi k_j (n-m)}{N}) h_j^*(m)$$

$$= \sum_{l=0}^{L-1} exp(\frac{-j2\pi k_i l}{N}) h_i(l) \sum_{m=0}^{L-1} exp(\frac{j2\pi k_j m}{N}) h_j^*(m) \sum_{n=0}^{N-1} a(k_i) a^*(k_j) exp(\frac{j2\pi (k_i - k_j) n}{N})$$

The first two terms in the above expression are constants and the third term follows the same orthogonality condition as in (1)